14.8 Lagrange Multipliers.

I. Find the local extreme values of a function $f(x,y,z)$, which are subject to a constraint $g(x,y,z)=0$. ($\nabla g \neq 0$)

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$$

II. If f is subject two constraints $g_1(x,y,z)$ and $g_2(x,y,z)$

$$\begin{cases} \nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \\ g_1 = 0 \\ g_2 = 0. \end{cases}$$

Thm 12 (Orthogonal Gradient Theorem)

Suppose f differentiable in a region whose interior contains a smooth curve $C: \vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$.

If P_0 is a pt on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to C at P_0.

Corollary. At the pts on a smooth curve $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j}$ where f takes on its local extremes relative to its values on the curve, we have $\nabla f \cdot \vec{v} = 0$, where $\vec{v} = \frac{d\vec{r}}{dt}$. ← velocity vector.

Remark. If f takes on a local extreme at P_0 on a surface $g=0$ Then f takes on a local extreme at P_0 on every differentiable curve through P_0 on the surface $g=0$.

Since ∇g is orthogonal to every velocity vector, and by Corollary, so does ∇f, then $\nabla f = \lambda \nabla g$.

Example: Let $a_1, \cdots, a_n$ be n positive numbers. Find the maximum of $\sum_{i=1}^{n} a_i x_i$ subject to the constraint $\sum_{i=1}^{n} x_i^2 = 1$.

Solution: $\nabla f = (a_1, \cdots, a_n)$, $\nabla g = (2x_1, \cdots, 2x_n)$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}, \quad \text{let } A = \sqrt{a_1^2 + \cdots + a_n^2},$$

then $\lambda = \frac{\pm A}{2}$ and $x_i = \frac{\pm a_i}{A}$.

$$f_{max} = \frac{1}{A} \sum a_i^2 = \sqrt{a_1^2 + \cdots + a_n^2}, \quad f_{min} = -\frac{1}{A} \sum a_i^2 = -\sqrt{a_1^2 + \cdots + a_n^2}.$$

Remark: Actually, it is the Cauchy inequality:

$$\left(\sum_{i=1}^{n} a_i x_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{i=1}^{n} a_i^2\right)$$